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# New scheme for pricing Bermudan options under stochastic volatility model

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**Abstract**

The author considers stochastic volatility models and introduces a new scheme for pricing Bermudan options under stochastic volatility models. His approach is the asymptotic expansion method which is based on Malliavin calculus.

## 1 Introduction

The valuation of Bermudan options is very important problem in option pricing theory. The values of Bermudan options in stochastic volatility models are calculated with the regression method developed by Longstaff and Schwartz [3]. This method is not suitable for parallel computing.

In this paper, we introduce a new scheme for pricing Bermudan options. This scheme is very universal and can be applied to problems we can not develop recombining trees. For example, we can apply to evaluations of derivatives under SV models.

Our scheme has two keys. One is to derive an approximate formula of the joint distribution function of stochastic processes using the asymptotic expansion method. The other is to develop recombining tree with the idea of binning [2] using the approximate joint distribution function. Using the recombining three, we evaluate derivatives like Bermudan options under stochastic volatility models. Our scheme is suitable for parallel computing.

The structure of this paper is as follows. The next section reviews the stochastic volatility models which are widely accepted in financial industry and applies the asymptotic expansion method to the model. The 3rd section describes how to derive our approximate formula of the joint distribution functions with the asymptotic expansion method, while the following section derives the joint distribution function of SABR model. The 5th section presents numerical results of our new scheme. The final section concludes.

# 2 Stochastic volatility model

## 2.1 Definition of stochastic volatility model

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\le t\le T}\right)$ be a complete probability space satisfying the usual hypotheses and $T \in (0,\infty)$ denotes some fixed horizon of economy. Let $(W_1(t), W_2(t))$, $0 \le t \le T$, be a 2-dimensional correlated Brownian motion with correlation given by $\rho : [0,T] \to [-1,1]$ such that

$$d\langle W_1, W_2\rangle_t = \rho(t)\, dt. \tag{1}$$

We consider the following stochastic differential equation for X and Y:

$$\begin{aligned} \mathrm{d}X(t) &= B(t, X(t), Y(t))\, \mathrm{d}W_1(t), & (2)\\ \mathrm{d}Y(t) &= M(t, Y(t))\, \mathrm{d}t + D(t, Y(t))\, \mathrm{d}W_2(t), & (3)\\ (X(0), Y(0)) &= (x_0, y_0) \in \mathbb{R}\times\mathbb{R}, & (4) \end{aligned}$$

Suppose $B$, $M$ and $D$ satisfy some regularity conditions.

## 2.2 Asymptotic expansion of stochastic volatility model

We consider an perturbed stochastic process defined as the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X^\epsilon(t) &= \epsilon B(t, X^\epsilon(t), Y^\epsilon(t))\, \mathrm{d}W_1(t), & (5)\\ \mathrm{d}Y^\epsilon(t) &= M(t, Y^\epsilon(t))\, \mathrm{d}t + \epsilon D(t, Y^\epsilon(t))\, \mathrm{d}W_2(t), & (6)\\ (X^\epsilon(0), Y^\epsilon(0)) &= (x_0, y_0) \in \mathbb{R}\times\mathbb{R}. & (7) \end{aligned}$$

We want to calculate an approximate solution of this model by using the asymptotic expansion approach. By results of [5], we have the following lemma.

**Lemma 2.1.** *$X^\epsilon(t)$ and $Y^\epsilon(t)$ have following approximate solutions as $\epsilon \to 0$ respectively.*

$$\begin{aligned} X^\epsilon(T) &= \sum_{i=0}^{N} \epsilon^i X_i(T)/i! + o\left(\epsilon^N\right), & (8)\\ Y^\epsilon(T) &= \sum_{i=0}^{N} \epsilon^i Y_i(T)/i! + o\left(\epsilon^N\right), & (9) \end{aligned}$$

*where*

$$X_i\,(T) = \left.\frac{\mathrm{d}^i X^\epsilon\,(T)}{\mathrm{d}\epsilon^i}\right|_{\epsilon=0}, \tag{10}$$

$$Y_i\,(T) = \left.\frac{\mathrm{d}^i Y^\epsilon\,(T)}{\mathrm{d}\epsilon^i}\right|_{\epsilon=0}, \tag{11}$$

*for* $i = 0, 1, \ldots, N$.

Here, we can calculate $X_i\,(T)$ and $Y_i\,(T)$ analytically. Examples of $Y_i\,(T)$ are as follows:

$$Y_1^\epsilon\,(T) \;=\; \tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, D\,(t_1, Y_0(t_1))\; \mathrm{d}W_2(t_1) \tag{12}$$

$$\begin{aligned} Y_2^\epsilon\,(T) \;&=\; \tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_1(t_1)^2\, M_{y,y}(t_1, Y_0(t_1))\, \mathrm{d}t_1, \\ &+\; 2\,\tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_1(t_1)\, D_y(t_1, Y_0(t_1))\, \mathrm{d}W_2(t_1) \end{aligned} \tag{13}$$

$$\begin{aligned} Y_3^\epsilon\,(T) \;&=\; \tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_1(t_1)^3\, M_{y,y,y}(t_1, Y_0(t_1))\, \mathrm{d}t_1, \\ &+\; 3\,\tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_1(t_1)\, Y_2(t_1)\, M_{y,y}(t_1, Y_0(t_1))\, \mathrm{d}t_1, \\ &+\; 3\,\tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_1(t_1)^2\, D_{y,y}(t_1, Y_0(t_1))\, \mathrm{d}W_2(t_1), \\ &+\; 3\,\tilde{M}\,(T) \int_0^T \tilde{M}\,(t_1)^{-1}\, Y_2(t_1)\, D_y(t_1, Y_0(t_1))\, \mathrm{d}W_2(t_1) \end{aligned} \tag{14}$$

where

$$\tilde{M}\,(T) \;=\; \exp\left(\int_0^T M_y(t_0, Y_0(t_0))\, \mathrm{d}t_0\right). \tag{15}$$

$$\tag{16}$$

And examples of $X_i(T)$ are as follows:

$$X_0^\epsilon(T) = x_0 \tag{17}$$

$$X_1^\epsilon(T) = \int_0^T B(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{18}$$

$$X_2^\epsilon(T) = 2\int_0^T X_1(t_o)\, B_y(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{19}$$

$$+ \; 2\int_0^T Y_1(t_o)\, B_x(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{20}$$

$$X_3^\epsilon(T) = 3\int_0^T X_1(t_o)^2\, B_{y,y}(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{21}$$

$$+ \; 3\int_0^T X_2(t_o)\, B_y(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{22}$$

$$+ \; 6\int_0^T X_1(t_o)\, Y_1(t_o)\, B_{x,y}(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{23}$$

$$+ \; 3\int_0^T Y_1(t_o)^2\, B_{x,x}(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{24}$$

$$+ \; 3\int_0^T Y_2(t_o)\, B_x(t_o, Y_0(t_o), X_0(t_o))\, \mathrm{d}W_1(t_o) \tag{25}$$

# 3 Approximation formula of the joint distribution function

We have to calculate conditional expectations to derive an approximate formula of the joint distribution function. The next theorem is very useful to calculate conditional expectations.

**Theorem 3.1.** *Let $f \in \mathbb{L}^2(\mathbb{T}^n)$ for $n \geq 1$, $q_1^j \in \mathbb{L}(\mathbb{T})$ for $1 \leq j \leq m$. Let $\{W_i\}_{i=1,\ldots,n}$ be an $n$-dimensional correlated Brownian motion and $\{Z_i\}_{i=1,\ldots,m}$ be an $m$-dimensional correlated Brownian motion. We denote $(t_1, t_2, \ldots, t_n)$ by $(\mathbf{t})$.*

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(\mathbf{t})\, \mathrm{d}W_n(t_n) \cdots \mathrm{d}W_2(t_2)\, \mathrm{d}W_1(t_1) \,\Bigg|\\
&\qquad \left\{\int_0^T q_1^1(t)\, \mathrm{d}Z_1(t), \ldots, \int_0^T q_1^m(t)\, \mathrm{d}Z_m(t)\right\} = \{c_1, \ldots, c_m\}\Bigg]\\
&= \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(\mathbf{t})\, \hat{H}_n(\mu(\mathbf{t}), \Sigma(\mathbf{t}))\, \mathrm{d}t_n \cdots \mathrm{d}t_2\, \mathrm{d}t_1,
\end{aligned} \tag{26}$$

*where*

$$
\begin{aligned}
\mathrm{d}\langle W_i, Z_j\rangle &= \rho_{i,j}\,\mathrm{d}t, && (27)\\
\Sigma_c &= \left\{\int_0^T q_i(t)\, q_j(t)\,\mathrm{d}t\right\}_{i,j=1,\ldots,m}, && (28)\\
\tilde{\Sigma}(\mathbf{t}) &= \{\rho_{i,j} q_j(t_i)\}_{i=1,\ldots,n,j=1,\ldots,m}, && (29)\\
\mu(\mathbf{t}) &= \Sigma_c^{-1\,t}\tilde{\Sigma}(\mathbf{t}), && (30)\\
\Sigma(\mathbf{t}) &= -\tilde{\Sigma}(\mathbf{t})\,\Sigma_c^{-1\,t}\tilde{\Sigma}(\mathbf{t}), && (31)\\
m(\xi; \mu(\mathbf{t}), \Sigma(\mathbf{t})) &= \exp\left(\mu(\mathbf{t})\,{}^t\xi + 1/2\xi\Sigma(\mathbf{t})\,{}^t\xi\right), && (32)\\
\tilde{H}_n(\mu(\mathbf{t}), \Sigma(\mathbf{t})) &= \left.\frac{\mathrm{d}^n m(\xi;\mu(\mathbf{t}),\Sigma(\mathbf{t}))}{\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n}\right|_{\xi=0}. && (33)
\end{aligned}
$$

Let $X_G^\epsilon(T) = (X^\epsilon(T) - X_0(T))/\epsilon$ and $Y_G^\epsilon(T) = (\sigma^\epsilon(T) - \sigma_0(T))/\epsilon$. We want to derive the joint distribution function of $X_G^\epsilon(T)$ and $Y_G^\epsilon(T)$. Let $\varphi_{X_G,Y_G} : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a characteristic function of $X_G^\epsilon(T)$ and $Y_G^\epsilon(T)$.

**Proposition 3.1.** *$\varphi_{X,Y}$ has an approximate expression as follows:*

$$
\varphi_{X_G,Y_G}(\xi_1,\xi_2) = \sum_{i=0}^{N}\frac{\epsilon^i}{i!}\left.\frac{\mathrm{d}^i\mathbb{E}\left[\exp\left(\sqrt{-1}\xi_1 X^\epsilon(T) + \sqrt{-1}\xi_2 Y^\epsilon(T)\right)\right]}{\mathrm{d}\epsilon^i}\right|_{\epsilon=0} + o\left(\epsilon^N\right) \tag{34}
$$

*for* $(\xi_1,\xi_2)\in\mathbb{R}\times\mathbb{R}$.

In case that $N=2$,

$$
\begin{aligned}
\varphi_{X_G,Y_G}(\xi_1,\xi_2) &= \mathbb{E}[N(T)] + \frac{\sqrt{-1}}{2}\mathbb{E}[(\xi_1 X_2(T) + \xi_2 Y_2(T))\,N(T)]\\
&\quad + \frac{\sqrt{-1}\epsilon}{6}\mathbb{E}[(\xi_1 X_3(T) + \xi_2 Y_3(T))\,N(T)]\\
&\quad - \frac{\epsilon^2}{8}\mathbb{E}\left[(\xi_1 X_2(T) + \xi_2 Y_2(T))^2 N(T)\right] + o\left(\epsilon^2\right) \qquad (35)
\end{aligned}
$$

where

$$
N(T) = \exp\left(\sqrt{-1}\xi_1 X_1(T) + \sqrt{-1}\xi_2 Y_1(T)\right). \tag{36}
$$

By using the inversion formulas of characteristic functions, we get an approximate formula of the joint probability density function of $X_G^\epsilon(T)$ and $Y_G^\epsilon(T)$.

**Proposition 3.2.** *$X_G^\epsilon(T)$ and $Y_G^\epsilon(T)$ have a 3rd order approximate joint prob-*

*ability density function $f_{X_G,Y_G}$ as follows:*

$$\begin{aligned} f_{X_G,Y_G}(x,y) &= n(x,y;\Sigma) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\{\mathbb{E}^c[X_2(T)]\,n(x,y;\Sigma)\} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}\{\mathbb{E}^c[Y_2(T)]\,n(x,y;\Sigma)\} \\ &\quad -\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}x}\{\mathbb{E}^c[X_3(T)]\,n(x,y;\Sigma)\} - \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}y}\{\mathbb{E}^c[Y_3(T)]\,n(x,y;\Sigma)\} \\ &\quad +\frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\mathbb{E}^c\left[X_2(T)^2\right]n(x,y;\Sigma)\right\} + \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\mathbb{E}^c\left[Y_2(T)^2\right]n(x,y;\Sigma)\right\} \\ &\quad +\frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x\,\mathrm{d}y}\{\mathbb{E}^c[X_2(T)\,n(x,y;\Sigma)]\,Y_2(T)\}, \end{aligned} \tag{37}$$

*for $(x,y) \in \mathbb{R}\times\mathbb{R}$, where*

$$\mathbb{E}^c[\cdot] = \mathbb{E}\left[\cdot\,\middle|\,(X_1(T),Y_1(T)) = (x,y)\right], \tag{38}$$

$$n(x,y;\Sigma) = \frac{1}{2\pi\sqrt{|\Sigma|}}\exp\left(-[x,y]\,\Sigma^{-1t}[x,y]\right), \tag{39}$$

$$\Sigma = \begin{bmatrix} \mathbb{E}\left[X_1(T)^2\right] & \mathbb{E}[X_1(T)\,Y_1(T)] \\ \mathbb{E}[X_1(T)\,Y_1(T)] & \mathbb{E}\left[Y_1(T)^2\right] \end{bmatrix}. \tag{40}$$

*Then, $X^\epsilon(T)$ and $Y^\epsilon(T)$ have a 3rd order approximate joint distribution function $F_{X,Y}$ as follows:*

$$F_{X,Y}(x,y) = \int_0^{x-X_0(T)}\int_0^{y-Y_0(T)} f_{X_G,Y_G}(v,w)\,\mathrm{d}w\,\mathrm{d}v \tag{41}$$

We can calculate conditional expectations in the above lemma by using Theorem 3.1.

# 4 Pricing Bermudan options

We introduce a new scheme for pricing Bermudan options under stochastic volatility models in this section. In order to clarify the dependency of the variables, we use notations as follows:

$$\begin{aligned} &F_{X,Y}(x_0,y_0,T,x,y) \\ &\quad = \mathbb{P}\left(X(T)\le x, Y(T)\le y \,\middle|\, X(0)=x_0, Y(0)=y_0\right). \end{aligned} \tag{42}$$

$$\begin{aligned} &\mathbb{P}_{X,Y}(x_0,y_0,T,l_x,u_x,l_y,u_y) \\ &\quad = \mathbb{P}\left(l_x \le X(T)\le u_x, l_y \le Y(T)\le u_y \,\middle|\, X(0)=x_0, Y(0)=y_0\right) \end{aligned} \tag{43}$$

We approximate $\mathbb{P}_{X,Y}(x_0,y_0,T,l_x,u_x,l_y,u_y)$ using results of Section 3. First, we have an approximate joint distribution function of $X$ and $Y$ by Proposition 3.2. Second, we calculate conditional expectations in the approximate joint distribution function using Theorem 3.1. Then we have an approximate formula of $\mathbb{P}_{X,Y}(x_0,y_0,T,l_x,u_x,l_y,u_y)$.

### 4.1 Bermudan options

Let $\mathbb{T}$ be $[T_0 = 0, T_1, T_2, \ldots, T_n, \infty]$ for $n \geq 1$ and $\mathcal{T}$ be a set of stopping time $\tau : \Omega \to \mathbb{T}$. We want to calculate a value $V(t)$ that is defined as follows:

$$V(t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[C(\tau, X(\tau), Y(\tau)) \middle| \mathcal{F}_t\right]. \tag{44}$$

We consider this option in this section.

### 4.2 New scheme

Let $\mathbb{X}$ be $[x_1, x_2, \ldots, x_N]$ and $\mathbb{Y}$ be $[y_1, y_2, \ldots, y_M]$ for $N \geq 1$ and $M \geq 1$ respectively. We define $a_i$ for $0 \leq i \leq N$ and $b_j$ for $0 \leq j \leq M$ as follows:

$$a_i = \begin{cases} -\infty & i = 0 \\ (x_i + x_{i+1})/2 & i = 1, 2, \ldots, N-1 \\ \infty & i = N \end{cases}, \tag{45}$$

$$b_i = \begin{cases} -\infty & i = 0 \\ (x_i + x_{i+1})/2 & i = 1, 2, \ldots, M-1 \\ \infty & i = M \end{cases}. \tag{46}$$

We calculate the value $V(k, i, j)$ of the option at time $T_k$ and $(X(T_k)\, Y(T_k)) = (x_i, y_j)$ as follows:
when $k = n$,

$$V(k, i, j) = C(T_k, x_i, y_j), \tag{47}$$

otherwise,

$$V(k, i, j) = \max\left(C(T_k, x_i, y_j), \sum_{\tilde{i}=1, \tilde{j}=1}^{N,M} V\left(k+1, \tilde{\imath}, \tilde{j}\right) \mathbb{P}\left(i, j, k+1, \tilde{i}, \tilde{j}\right)\right), \tag{48}$$

where

$$\mathbb{P}\left(i, j, k+1, \tilde{i}, \tilde{j}\right) = \mathbb{P}\left(x_i, y_i, T_{k+1} - T_k, a_{\tilde{i}-1}, a_{\tilde{i}}, b_{\tilde{j}-1}, b_{\tilde{j}}\right). \tag{49}$$

Derivatives are valued in this scheme by the usual backward induction method. Since a direct construction of a multidimensional tree would not lead to recombining nodes, the computational effort would grown exponentially in the number of time steps. However, the computational effort is $n \times N \times M$ in our scheme.

Table 1: Parameter

| | $x_0$ | $y_0$ | $\alpha$ | $\beta$ | $\rho$ | $\epsilon$ | $r$ |
|---|---|---|---|---|---|---|---|
| (i) | 100 | 0.3 | 0.3 | 1.0 | 0.2 | 1.0 | 0.01 |
| (ii) | 100 | 0.3 | 0.3 | 0.5 | 0.2 | 1.0 | 0.01 |

# 5 Numerical result

To test the validity of the new scheme, we consider Bermudan and European put option under the SABR model as follows:

$$\begin{aligned} \mathrm{d}X^{\epsilon}(t) &= \epsilon Y^{\epsilon}(t)\, X^{\epsilon}(t)^{\beta}\, \mathrm{d}W_1(t), && (50)\\ \mathrm{d}Y^{\epsilon}(t) &= \epsilon\alpha Y^{\epsilon}(t)\, \mathrm{d}W_2(t), && (51)\\ \mathrm{d}\langle X^{\epsilon}, Y^{\epsilon}\rangle_t &= \rho\, \mathrm{d}t, && (52)\\ (X^{\epsilon}(0), Y^{\epsilon}(0)) &= (x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^+, && (53)\\ S(T) &= \exp(rT)\, X^{\epsilon}(T). && (54) \end{aligned}$$

Let execution times of Bermudan options be $\mathbb{T} = \{1.0, 2.0, 3.0, 4.0\}$ and the maturity of European option be $T = 4.0$. We calculate following values.

$$\begin{aligned} \mathrm{Put}_{\mathrm{Eur}} &= \mathbb{E}\left[\exp(-rT)\,(K - S(T))^{+}\right], && (55)\\ \mathrm{Put}_{\mathrm{Ber}} &= \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\exp(-r\tau)\,(K - S(\tau))^{+}\right], && (56) \end{aligned}$$

where $\mathcal{T}$ is a set of stopping time $\tau : \Omega \to \mathbb{T}$ and $K$ is strike.

In the test of the new scheme, we set $N = 100$ and $M = 50$, and define $x_1$, $x_N$, $y_1$ and $y_M$ as follows:

$$\begin{aligned} x_1 &= \mathbb{E}[X^{\epsilon}(T)] + 5\mathbb{E}\left[(X^{\epsilon}(T) - X_0(T))^2\right]^{1/2} && (57)\\ x_N &= \mathbb{E}[X^{\epsilon}(T)] - 5\mathbb{E}\left[(X^{\epsilon}(T) - X_0(T))^2\right]^{1/2} && (58)\\ y_1 &= \mathbb{E}[X^{\epsilon}(T)] + 5\mathbb{E}\left[(X^{\epsilon}(T) - X_0(T))^2\right]^{1/2} && (59)\\ y_M &= \mathbb{E}[Y^{\epsilon}(T)] - 5\mathbb{E}\left[(X^{\epsilon}(T) - X_0(T))^2\right]^{1/2} && (60)\\ & && (61) \end{aligned}$$

The model parameters used in the test are given in Table 1. We use a 4th order asymptotic expansion for the joint distribution function and an approximate cumulative bivariate normal probabilities[1].

We use values which are calculated in Monte Carlo simulations as benchmarks. In the simulations, we use Ninomiya-Victoir scheme[4] as a discretization scheme with 8 time steps per a year and generate $10^7$ paths in each simulation.

Results are in Table 2. We compare our estimations of values by an asymptotic expansion with forth order to the bechmarks.

Table 2: Numerical results

| Case | Strike | Value $Put_{Ber}$ (A.E.) | Value $Put_{Eur}$ (A.E.) | Value $Put_{Eur}$ (M.C.) | Imp.Vol. $Put_{Eur}$ (A.E.) | Imp.Vol. $Put_{Eur}$ (M.C.) | Error (bpt) | Prob(ITM) |
|---|---|---|---|---|---|---|---|---|
| (i) | 50 | 2.9812 | 2.7406 | 2.6002 | 0.3220 | 0.3168 | 51.620 | 0.17 |
| (i) | 55 | 3.9393 | 3.6815 | 3.5629 | 0.3171 | 0.3136 | 35.840 | 0.22 |
| (i) | 60 | 5.1114 | 4.8334 | 4.7367 | 0.3137 | 0.3112 | 24.720 | 0.27 |
| (i) | 65 | 6.5411 | 6.2400 | 6.1324 | 0.3120 | 0.3096 | 23.860 | 0.31 |
| (i) | 70 | 8.1883 | 7.8583 | 7.7558 | 0.3106 | 0.3086 | 20.170 | 0.36 |
| (i) | 75 | 10.0547 | 9.6885 | 9.6075 | 0.3096 | 0.3082 | 14.460 | 0.41 |
| (i) | 80 | 12.1385 | 11.7289 | 11.6842 | 0.3089 | 0.3082 | 7.340 | 0.46 |
| (i) | 85 | 14.4529 | 13.9930 | 13.9778 | 0.3088 | 0.3085 | 2.330 | 0.50 |
| (i) | 90 | 17.0126 | 16.4966 | 16.4789 | 0.3095 | 0.3092 | 2.570 | 0.54 |
| (i) | 95 | 19.7618 | 19.1804 | 19.1749 | 0.3102 | 0.3102 | 0.760 | 0.58 |
| (i) | 100 | 22.6899 | 22.0339 | 22.0528 | 0.3111 | 0.3113 | -2.530 | 0.62 |
| (i) | 105 | 25.7867 | 25.0463 | 25.0984 | 0.3120 | 0.3127 | -6.820 | 0.65 |
| (i) | 110 | 29.0496 | 28.2181 | 28.2975 | 0.3131 | 0.3141 | -10.210 | 0.68 |
| (i) | 115 | 32.4811 | 31.5532 | 31.6370 | 0.3147 | 0.3157 | -10.640 | 0.71 |
| (i) | 120 | 36.0408 | 35.0061 | 35.1040 | 0.3162 | 0.3174 | -12.310 | 0.74 |
| (i) | 125 | 39.7158 | 38.5677 | 38.6868 | 0.3176 | 0.3191 | -14.930 | 0.76 |
| (i) | 130 | 43.4977 | 42.2293 | 42.3748 | 0.3191 | 0.3209 | -18.250 | 0.78 |
| (i) | 135 | 47.3857 | 45.9889 | 46.1573 | 0.3206 | 0.3227 | -21.180 | 0.80 |
| (i) | 140 | 51.3770 | 49.8482 | 50.0250 | 0.3224 | 0.3246 | -22.360 | 0.81 |
| (i) | 145 | 55.4463 | 53.7786 | 53.9694 | 0.3240 | 0.3265 | -24.320 | 0.83 |
| (i) | 150 | 59.5876 | 57.7742 | 57.9833 | 0.3257 | 0.3283 | -26.930 | 0.84 |
| (ii) | 90 | 0.0544 | 0.0526 | 0.0548 | 0.0359 | 0.0361 | -2.310 | 0.02 |
| (ii) | 92 | 0.0996 | 0.0944 | 0.0968 | 0.0344 | 0.0346 | -1.603 | 0.04 |
| (ii) | 94 | 0.1857 | 0.1697 | 0.1724 | 0.0331 | 0.0333 | -1.148 | 0.06 |
| (ii) | 96 | 0.3515 | 0.3050 | 0.3077 | 0.0321 | 0.0321 | -0.773 | 0.09 |
| (ii) | 98 | 0.6701 | 0.5431 | 0.5442 | 0.0313 | 0.0313 | -0.226 | 0.16 |
| (ii) | 100 | 1.2559 | 0.9389 | 0.9379 | 0.0308 | 0.0308 | 0.164 | 0.26 |
| (ii) | 102 | 2.2185 | 1.5514 | 1.5494 | 0.0307 | 0.0306 | 0.264 | 0.38 |
| (ii) | 104 | 3.5703 | 2.4210 | 2.4205 | 0.0308 | 0.0308 | 0.064 | 0.52 |
| (ii) | 106 | 5.2187 | 3.5534 | 3.5547 | 0.0314 | 0.0314 | -0.171 | 0.65 |
| (ii) | 108 | 7.0428 | 4.9166 | 4.9182 | 0.0322 | 0.0322 | -0.229 | 0.76 |
| (ii) | 110 | 8.9550 | 6.4566 | 6.4591 | 0.0331 | 0.0332 | -0.435 | 0.84 |
| (ii) | 112 | 10.9065 | 8.1231 | 8.1269 | 0.0342 | 0.0343 | -0.821 | 0.89 |
| (ii) | 114 | 12.8745 | 9.8757 | 9.8807 | 0.0353 | 0.0355 | -1.380 | 0.93 |
| (ii) | 116 | 14.8495 | 11.6856 | 11.6913 | 0.0365 | 0.0367 | -2.028 | 0.95 |
| (ii) | 118 | 16.8276 | 13.5330 | 13.5391 | 0.0377 | 0.0380 | -2.860 | 0.97 |
| (ii) | 120 | 18.8071 | 15.4045 | 15.4111 | 0.0388 | 0.0392 | -4.078 | 0.97 |

# A Proof of Theorem 3.1

## A.1 preliminaries

**Lemma A.1.** *Fixed $T \in (0,\infty)$. Let $\mathbb{T} = [0,T]$, $\mu$ be the Lebesgue measure, $f_n \in \mathbb{L}^2(\mathbb{T}^n, \sigma(\mathbb{T})^n, \mu^n)$ for $n \geq 1$ and $(W_1, W_2, \ldots, W_n)$ be a n-dimensional correlated Brownian motion. We denote by $\mathcal{E}_n$ the set of elementary functions of the form*

$$f(\mathbf{t}) = \sum_{i_1,\ldots,i_n=1}^{k} c_{i_1\cdots i_n} \mathbf{1}_{A_{i_1}\times\cdots\times A_{i_n}}(\mathbf{t}) \tag{62}$$

*where $A_1,\ldots,A_k$ are pairwise-disjoint sets belonging to $\sigma(\mathbb{T})$, and the coefficients $c_{i_1\cdots i_n}$ are zero if any two of the indices $i_1,\ldots,i_n$ are equal. Then there exists a sequence $\left\{f_n^{(l)}\right\}_{l\in\mathbb{N}} \in \mathcal{E}_n$ such that $f_n^{(l)} \nearrow f_n$ and*

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\, \mathrm{d}W_n(t_n)\cdots \mathrm{d}W_1(t_1)\middle|\mathcal{G}\right] \to \\ &\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n(\mathbf{t})\, \mathrm{d}W_n(t_n)\cdots \mathrm{d}W_1(t_1)\middle|\mathcal{G}\right] (a.s.),\end{aligned} \tag{63}$$

*where $\mathcal{G} \subset \sigma(\mathbb{T})$.*

## A.2 Proof

We use symbols in Lemma A.1. We set $\mathcal{G}$ as follows:

$$\mathcal{G} = \left\{\left(\int_0^T q_1^1(t)\, \mathrm{d}Z_1(t), \ldots, \int_0^T q_1^m(t)\, \mathrm{d}Z_m(t)\right) = (c_1,\ldots,c_m)\right\}. \tag{64}$$

Then we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\, \mathrm{d}W_n(t_n)\cdots \mathrm{d}W_1(t_1)\middle|\mathcal{G}\right] \\
=\ & \sum_{i_1,\ldots,i_n=1}^{k} c_{i_1\cdots i_n}\mathbb{E}\left[\int_0^T \mathbf{1}_{A_{i_n}}(t)\, \mathrm{d}W_n(t)\cdots \int_0^T \mathbf{1}_{A_{i_1}}(t)\, \mathrm{d}W_1(t)\middle|\mathcal{G}\right] \\
=\ & \int_0^T \cdots \int_0^T \sum_{i_1,\ldots,i_n=1}^{k} c_{i_1\cdots i_n}\mathbf{1}_{A_{i_1}\times\cdots\times A_{i_n}}(\mathbf{t})\, \hat{H}_n(\mu(\mathbf{t}), \Sigma(\mathbf{t}))\, \mathrm{d}t_n\cdots \mathrm{d}t_1 \\
=\ & \int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\, \hat{H}_n(\mu(\mathbf{t}), \Sigma(\mathbf{t}))\, \mathrm{d}t_n\cdots \mathrm{d}t_1 && (65)\\
\to\ & \int_0^T \cdots \int_0^T f_n(\mathbf{t})\, \hat{H}_n(\mu(\mathbf{t}), \Sigma(\mathbf{t}))\, \mathrm{d}t_n\cdots \mathrm{d}t_1 && (66)
\end{aligned}$$

We define $f_n(\mathbf{t})$ as follows:

$$f_n(\mathbf{t}) = \mathbf{1}_{\{t_n \leq \cdots \leq t_1\}}(\mathbf{t}) f(\mathbf{t}), \tag{67}$$

then we have Theorem 3.1.

# References


[1] Z. Drezner. Computation of the bivariate normal integral. *Mathematics of Computation*, 32(141):277–279, 1 1978.

[2] Christian P. Fries. Pricing options with early exercise by monte carlo simulation: An introduction. Working Paper, 2005.

[3] Francis A. Longstaff and Eduardo S. Schwartz. Valuing american options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14:113–147, 2001.

[4] Syoiti Ninomiya and Nicolas Victoir. Weak approximation of stochastic differential equations and application to derivative pricing. *Applied Mathematical Finance*, 15:107–121, 2008.

[5] Akihiko Takahashi, Kohta Takehara, and Masashi Toda. Computation in an asymptotic expansion method. Working Paper CIRJE-F-621, 2009.